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Existence of multiple positive periodic solutions for differential equation with state-dependent delays ☆

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Abstract

By utilizing a fixed point theorem in cones, we present some sufficient conditions which guarantee the existence of multiple positive periodic solutions for a class of differential equations with state-dependent delays. Our results extend and improve some previous results.

Keywords: State-dependent delay; Multiple positive periodic solution; Existence

1. Introduction

In this paper we study the state-dependent delay equations:

$$x'(t) = -a\big(t, x(t)\big)x(t) + f\Big(t, x\big(t-\tau_1\big(t, x(t)\big)\big), \dots, x\big(t-\tau_m\big(t, x(t)\big)\big)\Big), \tag{1}$$
$$x'(t) = a\big(t, x(t)\big)x(t) - f\Big(t, x\big(t-\tau_1\big(t, x(t)\big)\big), \dots, x\big(t-\tau_m\big(t, x(t)\big)\big)\Big), \tag{2}$$

where $a \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$, $a(t+\omega, x) = a(t, x)$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}^+$, $f \in C(\mathbb{R} \times [\mathbb{R}^+]^m, \mathbb{R}^+)$, $f(t+\omega, x_1, \dots, x_m) = f(t, x_1, \dots, x_m)$, $\tau_i(t+\omega, x) = \tau_i(t, x)$ for any $x \in \mathbb{R}^+$, $t \in \mathbb{R}$, $\mathbb{R}^+ = [0, \infty)$, $i = 1, 2, \dots, m$, $\omega > 0$ is a constant.

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In the case $m = 1$ and $\tau \equiv 1$, Eqs. (1) and (2) appear in several applications (see, e.g. [1–4] and references therein). Over the past several years it has become apparent that equations with state-dependent delays arise also in several areas such as in population models [5], in models of cell productions [6], and in models of commodity price fluctuations [7]. In the case $a(t,x) = a(t)$, $\tau_i(t, x(t)) = \tau_i(t)$, $i = 1, 2, \dots, m$, Liu [8] studied the existence of multiple positive periodic solutions of Eqs. (1), (2) by using the Krasnoselskii fixed point theorem. The main results of [8] are improved by the ones in this paper.

Some basic results on the existence, uniqueness, continuation and continuous dependence of the solutions for differential equations with state-dependent delays are obtained in [9,10]. On the existence of periodic solutions, Arino et al. [11], Li and Kuang [12] and Magal and Arino [13] gave some results of existence of periodic solution for differential equations with state-dependent delay. However, as far as we know, there is few papers discussing the existence of multiple positive periodic solutions for state-dependent delay differential equations.

Our goal in this paper is to represent conditions which guarantee the existence of multiple positive periodic solutions for Eqs. (1) and (2). By employing a fixed point theorem in cones we found that the existence of multiple positive solutions in periodic equations with state-dependent delays require only a set of natural and easily verifiable conditions.

In what follows, we only discuss the existence of positive periodic solutions of Eq. (1), though similar results can be obtained for Eq. (2). We always assume that

(A) $a_1(t) \leqslant a(t,x) \leqslant a_2(t)$ for any $(t,x) \in \mathbb{R} \times \mathbb{R}^+$, where a_1, a_2 are nonnegative ω-periodic continuous functions on $\mathbb{R}$ and $\int_0^\omega a_1(s)\,ds > 0$.

2. Main results and proof

Firstly, let us introduce the fixed point theorem in cones which will be used in this paper.

Lemma 1. [14] *Let $X = (X, \|\cdot\|)$ be a Banach space, let K be a cone in X, and let r_1 and r_2 be constants such that $0 < r_1 < r_2$. Suppose that $\Phi : \overline{\Omega}_{r_2} \cap K \to K$, where $\Omega_{r_2} = \{x \mid x \in X, \|x\| < r_2\}$, is a completely continuous operator satisfying the following conditions*:

(1) *$x \neq \lambda \Phi x$ for $x \in K \cap \partial\Omega_{r_1}$ and $\lambda \in [0, 1]$*;
(2) *there exists $\psi \in K \backslash \{0\}$ such that $x \neq \Phi x + \eta\psi$ for $x \in K \cap \partial\Omega_{r_2}$ and $\eta \geqslant 0$.*

Then Φ has a fixed point in $K \cap \{x \in X \mid r_1 < \|x\| < r_2\}$.

Lemma 2. [15] *The operator Φ has a fixed point in $K \cap \{x \in X \mid r_1 < \|x\| < r_2\}$ if conditions* (1) *and* (2) *in Lemma* 1 *are replaced by the following conditions*:

(1) *$x \neq \lambda \Phi x$ for $x \in K \cap \Omega_{r_2}$ and $\lambda \in [0, 1]$*;
(2) *there exists $\psi \in K \backslash \{0\}$ such that $x \neq \Phi x + \eta\psi$ for $x \in K \cap \partial\Omega_{r_1}$ and $\eta \geqslant 0$.*

Let $X = \{x(t) \mid x(t + \omega) = x(t),\ t \in \mathbb{R}\}$, and $\|x\| = \max_{x\in[0,\omega]} |x(t)|$. Then X is a Banach space when it is endowed with the norm $\|\cdot\|$.

Lemma 3. *x is a positive ω-periodic solution of Eq.* (1) *if and only if x is a ω-periodic solution of the integral equation*

$$x(t)=\int_t^{t+\omega} G(t,s)f\bigl(s,x\bigl(s-\tau_1\bigl(s,x(s)\bigr)\bigr),\dots,x\bigl(s-\tau_m\bigl(s,x(s)\bigr)\bigr)\bigr)\,ds, \tag{3}$$

where $G(t,s)=\frac{\exp(\int_t^s a(\xi,x(\xi))\,d\xi)}{\exp(\int_0^\omega a(t,x(t))\,dt)-1}$.

Proof. Suppose that x is a periodic solution of Eq. (1). Then multiplying Eq. (1) with $\exp(\int_0^t a(s,x(s))\,ds)$, we have

$$\begin{aligned}\frac{dx}{dt}\exp\left(\int_0^t a\bigl(s,x(s)\bigr)\,ds\right)={}&-a\bigl(t,x(t)\bigr)x(t)\exp\left(\int_0^t a\bigl(s,x(s)\bigr)\,ds\right)\\&+f\bigl(t,x\bigl(t-\tau_1\bigl(t,x(t)\bigr)\bigr),\dots,x\bigl(t-\tau_m\bigl(t,x(t)\bigr)\bigr)\bigr)\\&\times\exp\left(\int_0^t a\bigl(s,x(s)\bigr)\,ds\right).\end{aligned}$$

Integrating the above equation from t to $t+\omega$, we obtain

$$\begin{aligned}&\int_t^{t+\omega}\left(x(s)\exp\left(\int_0^s a\bigl(\eta,x(\eta)\bigr)\,d\eta\right)\right)'ds\\&\quad=x(t+\omega)\exp\left(\int_0^{t+\omega} a\bigl(s,x(s)\bigr)\,ds\right)-x(t)\exp\left(\int_0^t a\bigl(s,x(s)\bigr)\,ds\right)\\&\quad=\int_t^{t+\omega}\exp\left(\int_0^s a\bigl(\eta,x(\eta)\bigr)\,d\eta\right)f\bigl(s,x\bigl(s-\tau_1\bigl(s,x(s)\bigr)\bigr),\dots,x\bigl(s-\tau_m\bigl(s,x(s)\bigr)\bigr)\bigr)\,ds.\end{aligned}$$

Therefore we have

$$x(t)=\int_t^{t+\omega} G(t,s)f\bigl(s,x\bigl(s-\tau_1\bigl(s,x(s)\bigr)\bigr),\dots,x\bigl(s-\tau_m\bigl(s,x(s)\bigr)\bigr)\bigr)\,ds.$$

Thus, x is a periodic solution for Eq. (3).

If x is a periodic solution of Eq. (3), then deviating the two sides of Eq. (3) about t, we have that x is a periodic solution of Eq. (1). Thus we complete the proof. □

We define now an operator T on the Banach space X by

$$(Tx)(t)=\int_t^{t+\omega} G(t,s)f\bigl(s,x\bigl(s-\tau_1\bigl(s,x(s)\bigr)\bigr),\dots,x\bigl(s-\tau_m\bigl(s,x(s)\bigr)\bigr)\bigr)\,ds,$$
$$\forall x\in X,\ t\in\mathbb{R}.$$

Thus we have

$$\begin{aligned}
(Tx)(t+\omega) &= \int_{t+\omega}^{t+2\omega} G(t+\omega,s) f\big(s, x\big(s-\tau_1\big(s,x(s)\big)\big), \ldots, x\big(s-\tau_m\big(s,x(s)\big)\big)\big)\,ds \\
&= \int_{t}^{t+\omega} G(t,s) f\big(s, x\big(s-\tau_1\big(s,x(s)\big)\big), \ldots, x\big(s-\tau_m\big(s,x(s)\big)\big)\big)\,ds \\
&= (Tx)(t) \quad \text{for } t \in \mathbb{R},\ x \in X.
\end{aligned}$$

Hence $T: X \to X$. From the above analysis we have that x is a periodic solution of Eq. (1) if and only if x is a fixed point of the operator T in X.

Set

$$M_1 = \inf_{0 \leqslant t \leqslant s \leqslant \omega} \exp\left(\int_t^s a_1(\xi)\,d\xi\right), \qquad M_2 = \sup_{0 \leqslant t \leqslant s \leqslant \omega} \exp\left(\int_t^s a_2(\xi)\,d\xi\right),$$

$$K_1 = \exp\left(\int_0^\omega a_1(t)\,dt\right), \qquad K_2 = \exp\left(\int_0^\omega a_2(t)\,dt\right), \qquad \sigma = \frac{M_1(K_1-1)}{M_2(K_2-1)},$$

$$\gamma = \frac{K_2-1}{K_1-1}.$$

It is evident that $\sigma \in (0,1]$, $\gamma \geqslant 1$. We define a cone in X by

$$K = \big\{x \mid x \in X,\ x(t) \geqslant \sigma\|x\|\big\}.$$

Lemma 4. $T: K \to K$.

Proof. For any $x \in K$, $t \in \mathbb{R}$, $s \in [t, t+\omega]$, we have

$$\frac{M_1}{K_2-1} \leqslant \frac{\exp(\int_t^s a_1(\zeta)\,d\zeta)}{\exp(\int_0^\omega a_2(\zeta)\,d\zeta)-1} \leqslant G(t,s) \leqslant \frac{\exp(\int_t^s a_2(\zeta)\,d\zeta)}{\exp(\int_0^\omega a_1(\zeta)\,d\zeta)-1} \leqslant \frac{M_2}{K_1-1}.$$

Hence, for any $x \in X$, we have

$$\|Tx\| \leqslant \frac{M_2}{K_1-1} \int_0^\omega f\big(t, x\big(t-\tau_1\big(t,x(t)\big)\big), \ldots, x\big(t-\tau_m\big(t,x(t)\big)\big)\big)\,dt$$

and

$$\begin{aligned}
(Tx)(t) &\geqslant \frac{M_1}{K_2-1} \int_0^\omega f\big(t, x\big(t-\tau_1\big(t,x(t)\big)\big), \ldots, x\big(t-\tau_m\big(t,x(t)\big)\big)\big)\,dt \\
&\geqslant \frac{M_1(K_1-1)}{M_2(K_2-1)}\|Tx\| = \sigma\|Tx\|.
\end{aligned}$$

Therefore $Tx \in K$ for any $x \in X$. This completes the proof. □

Lemma 5. *$T: K \to K$ is completely continuous.*

Proof. We first show that T is continuous. Let Ω be an arbitrary open bounded set in K. There exists a number $M > 0$ such that $\|x\| \leqslant M$ for any $x \in \Omega$. $f(t, u_1, \dots, u_m)$ is uniformly continuous on $[0, \omega] \times [0, M]^m$ due to the continuity of $f(t, u_1, \dots, u_m)$ and periodicity of f respect to t. Therefore for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(t, u_1, \dots, u_m) - f(t, v_1, \dots, v_m)| < \epsilon$ for any $t \in \mathbb{R}$ and $u_i, v_i \in [0, M]$ such that $|u_i - v_i| < \delta$, $i = 1, 2, \dots, m$.

Choose an arbitrary point $x_0 \in \Omega$. Since $x_0(t)$ is continuous and periodic on $\mathbb{R}$, we have that $x_0(t)$ is uniformly continuous. Then there exists $\delta_1 > 0$ (choose $\delta_1 < \delta$) such that

$$\left|x_0(t_1) - x_0(t_2)\right| < \delta/4 \tag{4}$$

for any $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_1$.

With a similar deduction we have that $\tau_i(t, y)$, $i = 1, 2, \dots, m$, are uniformly continuous on $[0, \omega] \times [0, M]$ due to the continuity respect to (t, y) on $\mathbb{R} \times [0, M]$ and periodicity respect to t. Hence there exists a number $\delta_2 > 0$ (choose $\delta_2 < \delta_1/2$) such that

$$\left|\tau_i(t, u_1) - \tau_i(t, u_2)\right| < \delta_1 \tag{5}$$

for $t \in \mathbb{R}$ and $u_1, u_2 \in [0, M]$ with $|u_1 - u_2| < \delta_2$, $i = 1, 2, \dots, m$.

Hence, in view of (5) we have

$$\left|\tau_i\bigl(t, x_0(t)\bigr) - \tau_i\bigl(t, y(t)\bigr)\right| < \delta_1 \quad \text{for } t \in \mathbb{R},\ i = 1, 2, \dots, m, \tag{6}$$

for $\|x_0 - y\| < \delta_2$ and $y \in \Omega$.

It follows from (4)–(6) that

$$\begin{aligned}
&\left|x_0\bigl(t - \tau_i\bigl(t, x_0(t)\bigr)\bigr) - y\bigl(t - \tau_i\bigl(t, y(t)\bigr)\bigr)\right| \\
&\quad \leqslant \left|x_0\bigl(t - \tau_i\bigl(t, x_0(t)\bigr)\bigr) - x_0\bigl(t - \tau_i\bigl(t, y(t)\bigr)\bigr)\right| + \left|x_0\bigl(t - \tau_i\bigl(t, y(t)\bigr)\bigr) - y\bigl(t - \tau_i\bigl(t, y(t)\bigr)\bigr)\right| \\
&\quad \leqslant \delta/4 + \delta_2 < \delta/4 + \delta/2 < \delta, \quad i = 1, 2, \dots, m,
\end{aligned}$$

for $t \in \mathbb{R}$, $\|x_0 - y\| < \delta_2$ and $y \in \Omega$. Therefore, for any $y \in \Omega$, if $\|x_0 - y\| < \delta_2$ then

$$\begin{aligned}
&\bigl\| f\bigl(t, x_0\bigl(t - \tau_1\bigl(t, x_0(t)\bigr)\bigr), \dots, x_0\bigl(t - \tau_m\bigl(t, x_0(t)\bigr)\bigr)\bigr) \\
&\quad - f\bigl(t, y_0\bigl(t - \tau_1\bigl(t, y_0(t)\bigr)\bigr), \dots, y_0\bigl(t - \tau_m\bigl(t, y_0(t)\bigr)\bigr)\bigr)\bigr\| < \epsilon
\end{aligned}$$

for any $t \in \mathbb{R}$.

Hence, if $t \in \mathbb{R}$, $y \in \Omega$ and $\|x_0 - y\| < \delta_2$, we have

$$\begin{aligned}
&\left|(Tx_0)(t) - (Ty)(t)\right| \\
&\quad = \left| \int_t^{t+\omega} G(t, s)\bigl(f\bigl(s, x_0\bigl(s - \tau_1\bigl(s, x_0(s)\bigr)\bigr), \dots, x_0\bigl(s - \tau_m\bigl(s, x_0(s)\bigr)\bigr)\bigr) \right. \\
&\qquad \left. - f\bigl(s, y\bigl(s - \tau_1\bigl(s, y(s)\bigr)\bigr), \dots, y\bigl(s - \tau_m\bigl(s, y(s)\bigr)\bigr)\bigr)\bigr)\, ds \right| \\
&\quad \leqslant \frac{M_2\omega}{K_1 - 1}\epsilon.
\end{aligned}$$

Therefore, $\|Tx_0 - Ty\| \leqslant \frac{M_2\omega}{K_1-1}\epsilon$, i.e., the operator T is continuous at x_0. Thus T is continuous in Ω due to the arbitrariness of x_0 in Ω.

Next we show that $\{Tx \mid x \in \Omega\}$ is a family of uniformly bounded and equicontinuous functions on $[0, \omega]$. Since $f(t, u_1, \ldots, u_m)$ is bounded on $\mathbb{R} \times [0, M]^m$, there exists a number $M_3 > 0$ such that

$$\left\| f(t, u_1, \ldots, u_m) \right\| \leqslant M_3 \quad \text{for } t \in \mathbb{R},\ u_i \in [0, M],\ i = 1, 2, \ldots, m. \tag{7}$$

For any $x \in \Omega$, we have $\|x\| \leqslant M$ and

$$\begin{aligned} \left|(Tx)(t)\right| &= \left| \int_t^{t+\omega} G(t,s) f\big(s, x\big(s - \tau_1\big(s, x(s)\big)\big), \ldots, x\big(s - \tau_m\big(s, x(s)\big)\big)\big)\, ds \right| \\ &\leqslant \frac{M_2}{K_1 - 1} \omega M_1 \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

Hence

$$\|Tx\| \leqslant \frac{M_2}{K_1 - 1} \omega M_1. \tag{8}$$

Finally, for any $t \in \mathbb{R}$, we have

$$\frac{d(Tx)(t)}{dt} = a\big(t, x(t)\big)(Tx)(t) + f\big(t, x\big(t - \tau_1\big(t, x(t)\big)\big), \ldots, x\big(t - \tau_m\big(t, x(t)\big)\big)\big). \tag{9}$$

In view of (7)–(9), we obtain that

$$\left| \frac{d(Tx)(t)}{dt} \right| \leqslant a_2^* \frac{M_2 \omega M_3}{K_1 - 1} + M_1 =: M_4,$$

where $a_2^* = \max_{t \in [0,\omega]} |a_2(t)|$. Since M_1, M_2 are independent of x, we obtain that $\{Tx \mid x \in \Omega\}$ is a family of uniformly bounded and equicontinuous functions on $[0, \omega]$. By the theorem of Ascoli–Arzela, the operator T is a completely continuous. This proves the lemma. □

In this paper, we have the following conditions:

- (H_1) $\liminf_{|u| \to 0^+} \frac{f(t,u_1,\ldots,u_m)}{|u|} > \gamma a_1(t)$ and $\liminf_{|u| \to +\infty} \frac{f(t,u_1,\ldots,u_m)}{|u|} > \gamma a_1(t)$ uniformly for $t \in \mathbb{R}$;
- (H_2) $\limsup_{|u| \to 0^+} \frac{f(t,u_1,\ldots,u_m)}{|u|} < \frac{a_2(t)}{\gamma}$ and $\limsup_{|u| \to +\infty} \frac{f(t,u_1,\ldots,u_m)}{|u|} < \frac{a_2(t)}{\gamma}$ uniformly for $t \in \mathbb{R}$;
- (H_3) there exists a number $p > 0$ such that the inequality $\sigma p \leqslant |u| \leqslant p$ yields $f(t, u_1, \ldots, u_m) < a_2(t) \frac{p}{\gamma}$ for $t \in [0, \omega]$;
- (H_4) there exists a $p > 0$ such that the inequality $\sigma p \leqslant |u| \leqslant p$ yields $f(t, u_1, \ldots, u_m) > a_1(t) p \gamma$ for $t \in [0, \omega]$;

where $|u| = \max_i \{u_1, \ldots, u_m\}$.

Theorem 1. *Assume that hypotheses* (A), (H_1) *and* (H_3) *are true. Then Eq.* (1) *has at least two positive ω-periodic solutions x_1 and x_2 such that* $0 < \|x_1\| < p < \|x_2\|$.

Proof. According to the first inequality of (H_1), i.e., $\liminf_{|u| \to 0^+} \frac{f(t,u_1,\ldots,u_m)}{|u|} > \gamma a_1(t)$ uniformly for $t \in \mathbb{R}$, we can find a sufficiently small number $\epsilon > 0$ and a number r_1 with $0 < r_1 < p$ such that

$$f(t, u_1, \ldots, u_m) > \gamma a_1(t)(1 + \epsilon)|u| \quad \text{for } 0 < |u| < r_1,\ t \in [0, \omega]. \tag{10}$$

Hence, for $x \in K$ and $\|x\| = r_1$, we have $r_1 \geqslant x(t) \geqslant \sigma\|x\| = \sigma r_1 > 0$.

Put $\psi \equiv 1$. We now prove that

$$x \neq Tx + \eta\psi \quad \text{for } x \in K \cap \partial\Omega_{r_1} \text{ and } \eta \geqslant 0, \tag{11}$$

where $\Omega_{r_1} = \{x \in X \mid \|x\| < r_1\}$.

If not, then there exist $x_0 \in K \cap \partial\Omega_{r_1}$ and $\eta_0 \geqslant 0$ such that

$$x_0 = Tx_0 + \eta_0. \tag{12}$$

Let $\alpha = \min_{t\in[0,\omega]} x_0(t)$, then $\alpha > 0$. So, for $t \in \mathbb{R}$, from (10) to (12), we have

$$\begin{aligned}
x_0(t) &= (Tx_0)(t) + \eta_0 \\
&= \int_t^{t+\omega} G(t,s) f\big(s, x_0\big(s - \tau_1\big(s, x_0(s)\big)\big), \ldots, x_0\big(s - \tau_m\big(s, x_0(s)\big)\big)\big)\, ds + \eta_0 \\
&> \int_t^{t+\omega} G(t,s) a_1(s)(1+\epsilon)\gamma \max_{1\leqslant i\leqslant m} \big|x_0\big(s - \tau_i\big(s, x_0(s)\big)\big)\big|\, ds \\
&\geqslant (1+\epsilon)\alpha\gamma \int_t^{t+\omega} \frac{\exp(\int_t^s a_1(\xi)\, d\xi)}{\exp(\int_0^\omega a(\xi, x_0(\xi))\, d\xi) - 1} a_1(s)\, ds \\
&\geqslant (1+\omega)\alpha\gamma \frac{\exp(\int_0^\omega a_1(s)\, ds) - 1}{\exp(\int_0^\omega a_2(s)\, ds) - 1} \\
&= (1+\epsilon)\alpha,
\end{aligned}$$

which yields $\alpha > (1+\epsilon)\alpha$. It is a contradiction. Therefore (11) is valid.

Next, by using the inequality in (H_3) we prove that

$$x \neq \lambda Tx \quad \text{for any } x \in K \cap \partial\Omega_p,\ \lambda \in [0,1], \tag{13}$$

where $\Omega_p = \{x \in X \mid \|x\| < p\}$.

If not, then there exist $x_0 \in K \cap \partial\Omega_p$ and $\lambda_0 \in [0,1]$ such that

$$x_0 = \lambda_0 Tx_0. \tag{14}$$

Clearly, $\lambda_0 \neq 0$. If not, we would have $x_0 \equiv 0$, which contradicts that $x_0 \in K \cap \partial\Omega_p$. Thus, $\|x_0\| = p$ and $\sigma p \leqslant x_0(t) \leqslant p$ for $t \in \mathbb{R}$. By condition (H_3), we have

$$f\big(t, x_0\big(t - \tau_1\big(t, x_0(t)\big)\big), \ldots, x_0\big(t - \tau_m\big(t, x_0(t)\big)\big)\big) < \frac{a_2(t)p}{\gamma} \quad \text{for } t \in \mathbb{R}. \tag{15}$$

Then from (14) and (15), for $t \in \mathbb{R}$, we have

$$\begin{aligned}
x_0(t) &= \lambda_0(Tx_0)(t) \\
&= \lambda_0 \int_t^{t+\omega} G(t,s) f\big(s, x_0\big(s - \tau_1\big(s, x_0(s)\big)\big), \ldots, x_0\big(s - \tau_m\big(s, x_0(s)\big)\big)\big)\, ds \\
&< \int_t^{t+\omega} G(t,s) a_2(s) \frac{p}{\gamma}\, ds \leqslant \frac{p}{\gamma} \int_t^{t+\omega} \frac{\exp(\int_t^s a_2(\xi)\, d\xi)}{\exp(\int_0^\omega a_1(\xi)\, d\xi) - 1} a_2(t)\, ds
\end{aligned}$$

$$= \frac{p}{\gamma} \frac{\exp(\int_0^\omega a_2(s)\,ds) - 1}{\exp(\int_0^\omega a_1(s)\,ds) - 1} = \frac{p}{\gamma}\gamma = p,$$

which yields $\|x_0\| = p < p$. We arrive at a contradiction. Therefore (13) is valid.

In view of (11), (13) and Lemma 2, we have that T has a fixed point $x_1 \in K \cap \{x \mid r_1 < \|x\| < p\}$ and $x_1(t) \geqslant \sigma r_1 > 0$. Therefore $x_1(t)$ is a periodic positive solution of Eq. (1).

By the second inequality in (H_1), i.e., $\liminf_{|u|\to+\infty} \frac{f(t,u_1,\ldots,u_m)}{|u|} > \gamma a_1(t)$ uniformly for $t \in \mathbb{R}$, we can find a sufficiently small number $\epsilon > 0$ and a number $r_2 > p$ such that

$$f(t, u_1, \ldots, u_m) \geqslant a_1(t)(1+\epsilon)|u| \quad \text{for } |u| \geqslant r_2. \tag{16}$$

Set $\psi \equiv 1$. We show that

$$x \neq Tx + \eta\psi \quad \text{for } x \in K \cap \partial\Omega_{r_2} \text{ and } \eta \geqslant 0, \tag{17}$$

where $\Omega_{r_2} = \{x \in X \mid \|x\| < r_2\}$.

If not, then there exist $x_0 \in K \cap \partial\Omega_{r_2}$ and $\eta_0 \geqslant 0$ such that

$$x_0 = \Phi x_0 + \eta_0\psi. \tag{18}$$

Let $\beta = \min_{0\leqslant t\leqslant\omega} x_0(t)$, then $\beta > 0$. So, for $t \in \mathbb{R}$, from (16) and (18), we have

$$\begin{aligned}
x_0(t) &= (Tx_0)(t) + \eta_0 \\
&= \int_t^{t+\omega} G(t,s) f\big(s, x_0\big(s - \tau_1\big(s, x_0(s)\big)\big), \ldots, x_0\big(s - \tau_m\big(s, x_0(s)\big)\big)\big)\,ds + \eta_0 \\
&\geqslant \int_t^{t+\omega} G(t,s)\gamma a_1(s)(1+\epsilon) \max_{1\leqslant i\leqslant m}\big\{x_0\big(s - \tau_i\big(s, x_0(s)\big)\big)\big\}\,ds \\
&\geqslant (1+\epsilon)\beta\gamma \int_t^{t+\omega} \frac{\exp(\int_t^s a_1(\xi)\,d\xi)}{\exp(\int_0^\omega a_2(\xi)\,d\xi) - 1} a_1(t)\,ds \\
&= (1+\epsilon)\beta\frac{\gamma}{\gamma} = (1+\epsilon)\beta.
\end{aligned}$$

Hence $\beta \geqslant (1+\epsilon)\beta$, which yields a contradiction. Thus (17) is valid.

In view of (13), (17) and Lemma 1, we obtain that T has a fixed point $x_2 \in K \cap \{x \mid p < \|x\| < r_2\}$ and $x_2(t) \geqslant \sigma p > 0$. Thus, $x_2(t)$ is a positive ω-periodic solution of Eq. (1). Therefore Eq. (1) has at least two positive periodic solutions. The proof is completed. □

Obviously, from Theorem 1 we have:

Corollary 1. *The conclusion of Theorem* 1 *remains valid if conditions* (A), (H_3) *are true and* (H_1) *is replaced by the following conditions*:

(H_1^*) $$\lim_{|u|\to 0^+} \min_{t\in[0,\omega]} \frac{f(t,u_1,\ldots,u_m)}{|u|} = +\infty \quad \textit{and} \quad \lim_{|u|\to+\infty} \min_{t\in[0,\omega]} \frac{f(t,u_1,\ldots,u_m)}{|u|} = +\infty.$$

Remark 1. Theorem 1 extends and improves Theorem 2.1 of [8] in the sense that, not only condition (H_1) of Theorem 1 is weaker than that of Theorem 2.1 of [8], i.e., the limits need not to be $+\infty$, but also that if Eq. (1), even the delays are not state-dependent, is transformed into

$x'(t) = -a_1(t)x(t) + f_1(t, x(t-\tau_1(t)), \ldots, x(t-\tau_m(t)))$, maybe the function f_1 is not positive (see Example 1). So Theorem 2.1 does not apply to such equations.

Theorem 2. *Suppose that conditions* (A), (H_2) *and* (H_4) *are satisfied. Then Eq.* (1) *has at least two periodic positive solutions* x_1 *and* x_2 *such that* $0 < \|x_1\| < p < \|x_2\|$.

Proof. By the first inequality in (H_2), that is, $\limsup_{|u|\to 0^+} \frac{f(t,u_1,\ldots,u_m)}{|u|} < \frac{a_2(t)}{\gamma}$ uniformly for $t \in \mathbb{R}$, we can obtain a sufficient small number $\epsilon > 0$ and a number r_1 with $0 < r_1 < p$ such that

$$f(t, u_1, \ldots, u_m) < \frac{a_2(t)}{\gamma}(1-\epsilon)|u| \quad \text{for } |u| \leqslant r_1,\ t \in [0, \omega]. \tag{19}$$

Hence, $r_1 \geqslant x(t) \geqslant \sigma r_1$ for $x \in K \cap \partial\Omega_{r_1}$, where $\Omega_{r_1} = \{x \in X \mid \|x\| < r_1\}$. Next we show that

$$x \neq \lambda T x \quad \text{for } x \in K \cap \Omega_{r_1} \text{ and } \lambda \in [0, 1]. \tag{20}$$

If not, there exist $x_0 \in K \cap \partial\Omega_{r_1}$ and $\lambda_0 \in [0, 1]$ such that

$$x_0 = \lambda_0 T x_0. \tag{21}$$

Clearly, $\lambda_0 \neq 0$. If not, we would have $x_0 \equiv 0$, which contradicts that $x_0 \in K \cap \partial\Omega_{r_1}$.

Let $\alpha = \max_{t\in[0,\omega]} x_0(t)$, then $\alpha > 0$. Since $\sigma r_1 \leqslant x_0(t) \leqslant r_1$, then for $t \in \mathbb{R}$, from (19) and (21), we have

$$\begin{aligned}
x_0(t) &= \lambda_0 (T x_0)(t) \\
&= \lambda_0 \int_t^{t+\omega} G(t,s) f\big(s, x_0\big(s - \tau_1\big(s, x_0(s)\big)\big), \ldots, x_0\big(s - \tau_m\big(s, x_0(s)\big)\big)\big)\, ds \\
&< \int_t^{t+\omega} G(t,s) \frac{a_2(s)}{\gamma}(1-\epsilon) \max_{1\leqslant i\leqslant m} \big|x_0\big(s - \tau_i\big(s, x_0(s)\big)\big)\big|\, ds \\
&< \frac{\alpha(1-\epsilon)}{\gamma} \int_t^{t+\omega} \frac{\exp(\int_t^s a_2(\eta)\, d\eta)}{\exp(\int_0^\omega a_1(\eta)\, d\eta) - 1} a_2(s)\, ds \\
&= \alpha(1-\epsilon)\frac{\gamma}{\gamma} = \alpha(1-\epsilon).
\end{aligned}$$

So we obtain $\alpha < \alpha(1-\epsilon)$, which is a contradiction. Therefore (20) is valid.

By utilizing the inequality in (H_4), we now prove that

$$x \neq T x + \eta\psi \quad \text{for } x \in K \cap \partial\Omega_p \text{ and } \eta \geqslant 0, \tag{22}$$

where $\Omega_p = \{x \in X \mid \|x\| < p\}$.

Set $\psi \equiv 1$. If (22) is not satisfied then there would exist $x_0 \in K \cap \partial\Omega_p$ and $\eta_0 \geqslant 0$ such that

$$x_0 = T x_0 + \eta\psi. \tag{23}$$

Since $x_0 \in K \cap \partial\Omega_p$, then $\|x\|_0 = p$, $p \geqslant x_0(t) \geqslant \sigma\|x_0\| = \sigma p$. By ($H_4$), for $t \in \mathbb{R}$, we have

$$f\big(t, x_0\big(t - \tau_1\big(t, x_0(t)\big)\big), \ldots, x_0\big(t - \tau_m\big(t, x_0(t)\big)\big)\big) > a_1(t) p\gamma. \tag{24}$$

Therefore, for $t \in [0, \omega]$, from (23) and (24), we have

$$\begin{aligned}
x_0(t) &= (Tx_0)(t) + \eta_0 \\
&\geqslant \int_t^{t+\omega} G(t,s) f\big(s, x_0\big(s - \tau_1\big(s, x_0(s)\big)\big), \ldots, x_0\big(s - \tau_m\big(s, x_0(s)\big)\big)\big)\, ds \\
&> \int_t^{t+\omega} G(t,s) a_1(t) p\gamma\, ds \geqslant p\gamma \int_t^{t+\omega} \frac{\exp(\int_t^s a_1(\xi)\, d\xi)}{\exp(\int_1^\omega a_2(\xi)\, d\xi) - 1} a_1(s)\, ds \geqslant p.
\end{aligned}$$

Thus we obtain $p > p$, which is a contradiction. Therefore (22) is valid.

In view of (20), (22) and Lemma 1, we obtain that T has a fixed point $x_1 \in K$ such that $r_1 < \|x_1\| < p$ and $x_1(t) \geqslant \sigma r_1 > 0$. Thus x_1 is a periodic positive solution of Eq. (1).

Secondly, by the second inequality in (H_2), i.e., $\limsup_{|u| \to +\infty} \frac{f(t,u_1,\ldots,u_m)}{|u|} < \frac{a_2(t)}{\gamma}$ uniformly for $t \in \mathbb{R}$, we can find a sufficiently small number $\epsilon > 0$ and a number $r_2 > p$ such that

$$f(t, u_1, \ldots, u_m) < \frac{a_2(t)}{\gamma}(1-\epsilon)|u| \quad \text{for } |u| \geqslant r_2,\ t \in \mathbb{R}. \tag{25}$$

Set $r_3 = r_2/\sigma$. Hence $x(t) \geqslant \sigma\|x\| = \sigma r_3 = r_2$ for any $x \in K \cap \partial\Omega_{r_3}$, where $\Omega_{r_3} = \{x \in X \mid \|x\| < r_3\}$.

We now show that

$$x \neq \lambda T x \quad \text{for } x \in K \cap \partial\Omega_{r_3} \text{ and } \lambda \in [0,1]. \tag{26}$$

If not, there exist $x_0 \in K \cap \partial\Omega_{r_3}$ and $\lambda_0 \in [0,1]$ such that

$$x_0 = \lambda_0 T x_0. \tag{27}$$

Evidently, $\lambda_0 \neq 0$. If not, then $x_0 \equiv 0$, which contracts that $x_0 \in K \cap \partial\Omega_{r_3}$. Thus for $t \in \mathbb{R}$, from (25) and (27), we have

$$\begin{aligned}
x_0(t) &= \lambda_0 (Tx_0)(t) \\
&= \lambda_0 \int_t^{t+\omega} G(t,s) f\big(s, x_0\big(s - \tau_1\big(s, x_0(s)\big)\big), \ldots, x_0\big(s - \tau_m\big(s, x_0(s)\big)\big)\big)\, ds \\
&\leqslant \int_t^{t+\omega} G(t,s) \frac{a_1(s)}{\gamma}(1-\epsilon) \max_{1 \leqslant i \leqslant m} \big|x_0\big(s - \tau_i\big(s, x_0(s)\big)\big)\big|\, ds \\
&\leqslant \frac{1-\epsilon}{\gamma} r_3 \int_t^{t+\omega} \frac{\exp(\int_t^s a_2(\xi)\, d\xi)}{\exp(\int_0^\omega a_1(\xi)\, d\xi) - 1} a_2(s)\, ds \\
&= \frac{1-\epsilon}{\gamma} \gamma r_3 = (1-\epsilon) r_3.
\end{aligned}$$

Therefore we have $r_3 \leqslant (1-\epsilon) r_3$, which is a contradiction. Thus (26) is valid.

In view of (22), (26) and Lemma 2, we have that the operator T has a fixed point $x_2 \in K \cap \{x \mid p < \|x\| < r_3\}$ and $x_2(t) \geqslant \sigma r_3 > 0$. Therefore x_2 is a positive ω-periodic solution of Eq. (1). This completes the proof of this theorem. □

Evidently, from Theorem 2 we have:

Corollary 2. *Theorem* 2 *is valid if conditions* (A) *and* (H_4) *are true and* (H_2) *is replaced by the following condition*:

$$(\mathrm{H}_2^*)\qquad \lim_{|u|\to 0^+}\max_{t\in[0,\omega]}\frac{f(t,u_1,\ldots,u_m)}{|u|}=0 \quad \textit{and} \quad \lim_{|u|\to+\infty}\max_{t\in[0,\omega]}\frac{f(t,u_1,\ldots,u_m)}{|u|}=0.$$

Remark 2. Theorem 2 extends and improves Theorem 2.2 of [8] in the sense that, not only condition (H_2) of Theorem 2 is weaker than condition (H_3) of Theorem of [8], i.e., the limits need not to be 0, but also that if Eq. (1), even the delays are not state-dependent, is transformed into $x'(t)=a_1(t)x(t)-f_1(t,x(t-\tau_1(t)),\ldots,x(t-\tau_m(t)))$, maybe the function f_1 is not positive. So Theorem 2.2 does not apply to such equations.

Corollary 3. *If one of the following conditions*:

(1) $\liminf_{|u|\to 0^+}\frac{f(t,u_1,\ldots,u_m)}{|u|}>\gamma a_1(t)$ *uniformly for* $t\in\mathbb{R}$ *and there exists* $p>0$ *such that the inequality* $\sigma p\leqslant |u|\leqslant p$ *yields* $f(t,u_1,\ldots,u_m)<a_2(t)\frac{p}{\gamma}$ *for* $t\in[0,\omega]$;
(2) $\liminf_{|u|\to +\infty}\frac{f(t,u_1,\ldots,u_m)}{|u|}>\gamma a_1(t)$ *uniformly for* $t\in\mathbb{R}$ *and there exists* $p>0$ *such that the inequality* $\sigma p\leqslant |u|\leqslant p$ *yields* $f(t,u_1,\ldots,u_m)<a_2(t)\frac{p}{\gamma}$ *holds for* $t\in[0,\omega]$;
(3) $\liminf_{|u|\to 0^+}\frac{f(t,u_1,\ldots,u_m)}{|u|}>\gamma a_1(t)$ *and* $\limsup_{|u|\to+\infty}\frac{f(t,u_1,\ldots,u_m)}{|u|}<\frac{a_2(t)}{\gamma}$ *uniformly for* $t\in\mathbb{R}$;
(4) $\limsup_{|u|\to 0^+}\frac{f(t,u_1,\ldots,u_m)}{|u|}<\frac{a_2(t)}{\gamma}$ *uniformly for* $t\in\mathbb{R}$ *and there exists* $p>0$ *such that* $\sigma p\leqslant |u|\leqslant p$ *yields* $f(t,u_1,\ldots,u_m)>a_1(t)p\gamma$ *for* $t\in[0,\omega]$;
(5) $\limsup_{|u|\to +\infty}\frac{f(t,u_1,\ldots,u_m)}{|u|}<\frac{a_2(t)}{\gamma}$ *uniformly for* $t\in\mathbb{R}$ *and there exists* $p>0$ *such that* $\sigma p\leqslant |u|\leqslant p$ *yields* $f(t,u_1,\ldots,u_m)>a_1(t)p\gamma$ *for* $t\in[0,\omega]$;
(6) $\limsup_{|u|\to 0^+}\frac{f(t,u_1,\ldots,u_m)}{|u|}<\frac{a_2(t)}{\gamma}$ *and* $\liminf_{|u|\to+\infty}\frac{f(t,u_1,\ldots,u_m)}{|u|}>\gamma a_1(t)$ *uniformly for* $t\in\mathbb{R}$,

holds then Eq. (1) *has at least one positive periodic solution.*

Proof. From the proof of Theorems 1 and 2 we can see that the conclusions of this corollary is valid. We shall omit it here. □

Example 1. We now consider the equation

$$\begin{aligned} x'(t)={}&-\left(1+\sin^2 t\right)\left(1+\frac{1}{1+x(t)}\right)x(t)+b\big[x^\alpha\big(t-\tau_1\big(t,x(t)\big)\big)\\ &+x^\beta\big(t-\tau_1\big(t,x(t)\big)\big)+x^\alpha\big(t-\tau_2\big(t,x(t)\big)\big)+x^\beta\big(t-\tau_2\big(t,x(t)\big)\big)\big], \end{aligned}$$

where the constant $0<b\leqslant(4+4e^{3\pi})^{-1}$; the constants α and β satisfy $0<\alpha<1<\beta$; $\tau_i(t,s)\in C(\mathbb{R}\times\mathbb{R}^+,\mathbb{R})$, $\tau(t+\pi,s)=\tau_i(t,s)$ for any $(t,s)\in\mathbb{R}\times\mathbb{R}^+$, $i=1,2$.

Because $a(t,x)=(1+\sin^2 t)(1+\frac{1}{1+x})$, taking $a_1(t)=1+\sin^2 t$, $a_2(t)=2(1+\sin^2 t)$, then we have that

$$a_1(t)\leqslant a(t,x)\leqslant a_2(t),$$

$$K_1 = \exp\left(\int_0^{\pi} \left(1+\sin^2 t\right) dt\right) = e^{3\pi/2}, \qquad K_2 = \exp\left(\int_0^{\pi} 2\left(1+\sin^2 t\right) dt\right) = e^{3\pi},$$

$$\gamma = \frac{K_2 - 1}{K_1 - 1} = e^{3\pi/2} + 1, \qquad \sigma = \frac{K_1 - 1}{K_2(K_2 - 1)} = \left[e^{3\pi}\left(e^{3\pi/2} + 1\right)\right]^{-1}.$$

Thus condition (A) is valid.

Because $f(t, u_1, u_2) = b[u_1^\alpha + u_1^\beta + u_2^\alpha + u_2^\beta]$, then we have that

$$\lim_{|u|\to 0^+} \min_{t\in[0,\omega]} \frac{f(t, u_1, u_2)}{|u|} = +\infty, \qquad \lim_{|u|\to +\infty} \min_{t\in[0,\omega]} \frac{f(t, u_1, u_2)}{|u|} = +\infty$$

and condition (H_1^*) is valid. Moreover, for $p = 1$, when $\sigma p \leqslant u_i \leqslant p$, $i = 1, 2$, we have

$$f(t, u_1, u_2) \leqslant 4b \leqslant \frac{1}{1+e^{3\pi}} < \frac{1}{1+e^{3\pi/2}} < 2\left(1+\sin^2 t\right)\frac{1}{1+e^{3\pi/2}} = a_2(t)\frac{p}{\gamma}, \quad t \in \mathbb{R}.$$

Thus condition (H_3) is satisfied. By Corollary 1, this equation has at least two positive π-periodic solutions. But if the equation, even the delays are not state-dependent, is transformed into

$$\begin{aligned} x'(t) = {} & -\left(1+\sin^2 t\right)x(t) + b\left[x^\alpha\left(t - \tau_1(t)\right) + x^\beta\left(t - \tau_1(t)\right)\right. \\ & \left. + x^\alpha\left(t - \tau_2(t)\right) + x^\beta\left(t - \tau_2(t)\right)\right] - \left(1+\sin^2 t\right)\frac{x(t)}{1+x(t)}, \end{aligned}$$

then

$$\begin{aligned} f = {} & b\left[x^\alpha\left(t - \tau_1(t)\right) + x^\beta\left(t - \tau_1(t)\right) + x^\alpha\left(t - \tau_2(t)\right) + x^\beta\left(t - \tau_2(t)\right)\right] \\ & - \left(1+\sin^2 t\right)\frac{x(t)}{1+x(t)} \end{aligned}$$

is not positive since $b > 0$ can be sufficiently small. So Theorem 2.1 in [8] does not apply to such equations.

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